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ON THE FREE EXPANSION OF TWO-DIMENSIONAL STREAMS OF PERFECT GAS*

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The laws of free expansion of plane and axisymmetric streams of perfect gas are investigated using the model of continuous medium under conditions of discharge into vacuum, at the initial sections of streams flowing into a region of low but finite pressure, and also in the nozzle accelerating section.

It is shown that when the adiabatic exponent $\varkappa$ does not exceed 2 and 3/2 in the plane and axisymmetric case, respectively, the conclusion reached by Ladyzhenskii /1—6/ on the rectilinearity of streamlines in an "ideal field" of the stream, and the formulas in /1—6/ for the flow parameters need more precise definition. In the plane case such refinement, using hodograph variables, made it possible to obtain expressions for the stream parameters that ensure good precision not only at high but, also, at moderate Mach numbers. By increasing the number of terms of these expressions or extending the initial region of the "near field" in numerical calculations (coefficients of these expressions are obtained using the conditions of merging within and at the boundary of that region), it is possible to infinitely increase the exactness of formulas that are valid at, as far as desired, distances from the initial stream cross section. Similar formulas are obtained also for the axisymmetric case, although the analysis here is not so rigorous, since the passing to the hodograph plane is preceded by the simplification of the continuity equation which is valid only near the axis of symmetry. The regularity of derived formulas for the plane, as well as for the axisymmetric case was confirmed by comparing the laws that follow from them with similar laws based on extensive numerical calculations effected by the method of characteristics.

1. Let us consider an axisymmetric or plane stream of perfect (inviscid and non-heat-conducting) gas, freely flowing from a convergent or convergent-divergent nozzle. We direct the x -axis of a Cartesian (xyz) or cylindrical $(xy\varphi)$ coordinate system downstream (from left to right) and locate the plane $x=0$ at the minimum (or, if specifically stated, at the outlet) nozzle cross section. In the plane case we shall consider only nozzles that have a plane of symmetry, although this restriction is, in fact, not essential. We identify that plane with the xz-plane and, as in the axisymmetric case, consider only values $y \geqslant 0$. By definition, the parameters of the investigated flows depend only on x and y, but the velocity vector $\mathbf{q}$ may have all its three components u, v and w nonzero. Below, free expansion is to be understood as the flow into vacuum that obtains either at the wall discontinuity at point a when the magnitude of the latter exceeds some limit value, or along a wall whose angle of inclination to the x-axis continuously increases with $x > x_a = 0$ thus ensuring the expansion of gas to zero pressure. Both possibilities are shown in Fig.1 in which the nozzle wall is shaded and the dot line represents the sonic line. Figure 1,a corresponds to a break in the wall and 1,b to a wall formed by an arc of radius r. When $r=0$ the second case reduces to the first.

The "parabolic degeneration" to which attention was evidently first drawn by Ladyzhenskii /1/, is an important property of free expansion flows. This property consists of the following. The considered flow is defined, in addition to streamlines along which the total enthalpy,

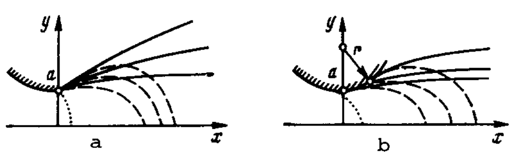

Fig.1

entropy, and "circulation" $\Gamma \equiv y^\nu w$ with $\nu=0$ and 1, respectively, in the plane and axisymmetric cases, and with supersonic "meriodional" component $\mathbf{V}$ of vector $\mathbf{q}$ has two sets of real characteristics (c^+ and c^- characteristics or characteristics of the first and second set),

*Prikl.Matem.Mekhan.,44,No.2,271-280,1980 along which $$\frac{dy}{dx} = \operatorname{tg}(\vartheta \pm \alpha), \quad \frac{d\vartheta}{dy} \pm \frac{\operatorname{ctg}\alpha}{\rho V^2}\frac{dp}{dy} \pm \frac{\nu}{y}\left[\frac{(a^2 + w^2)\sin\vartheta \sin\alpha}{a^2 \sin(\vartheta \pm \alpha)} - \frac{w^2 \operatorname{ctg}\alpha}{V^2}\right] = 0 \qquad (1.1)$$

where the upper (lower) signs correspond to c^+ (c^-) characteristics, ϑ is the angle of inclination of $\mathbf{V}$ to the x-axis, $V = |\mathbf{V}| = \sqrt{u^2 + v^2}$. The Mach angle α is determined by the meridional Mach number $M = V/a$, where a is the speed of sound, ρ is the density and p the pressure. In (1.1) and below all variables are dimensionless quantities, with those of dimension length normalized with respect to y_*° of the ordinate of the wall at minimum cross section, and those of dimensions of velocity, density, and pressure normalized with respect to a_*°, ρ_*° and $\rho_*^\circ a_*^{\circ 2}$, where a_*° and ρ_*° are some constants of dimension velocity and density. In calculations of isoenergetic and isentropic flows the critical velocity and density were taken for a_*° and ρ_*°.

In the case of free expansion at the corner or along the wall (when $r > 0$) angle ϑ continuously increases, while α decreases and approaches zero. Hence there exists a characteristic c^- whose angle of inclination to the wall is so small that it intersects streamlines and reaches the x-axis only when $x \to \infty$. The next following characteristics c^- do not, generally, reach the x-axis. The investigation presented below shows that the described situation actually obtains, if $\varkappa$ is not very close to unity. Let $\Sigma_0, \Sigma_1, \ldots$ be segments of isobars between the wall and the x-axis, whose initial points coincide with the initial points of the indicated characteristics. In Fig.1 $\Sigma_0, \Sigma_1, \ldots$ are denoted by dash lines and the c^--characteristics by thin continuous lines. By virtue of the above the region of definiteness of Ω_i of each segment Σ_i stretches along the x-axis, and for $\Sigma_1, \ldots$ also along y to infinity, with the parameters in Ω_i determined by the condition of symmetry $\vartheta = 0$ at $y = 0$ and the input data on Σ_i. The similarity of the described situation with that occuring with parabolic type equations explains the term "parabolic degeneration".

The absence of any conditions at the upper bounds of Ω_i substantially simplifies the derivation of solution, and this will be utilized below. Before proceeding any further, we shall present the results of calculations related to our problem, particularly as regards isoenergetic isentropic and irrotational $(w \equiv 0)$ streams at $\vartheta \equiv 0$ and $M \equiv M_0 \simeq 1$ in the $x = 0$

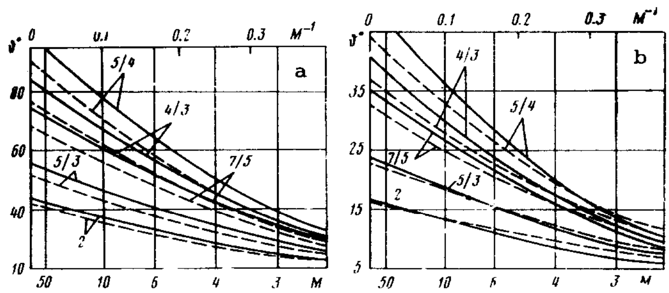

Fig.2

cross section. When $\nu = 0$, $r = 0$ then in conformity with (1.1), the dependence established by the c^--characteristics connecting the respective points of the x-axis with those of the characteristics in the beam at a, between the Mach number M at the stream axis and angle ϑ in the beam, is of the form

$$\vartheta = \theta(M, M_0) = \frac{1}{2}[\Phi(M) - \Phi(M_0)], \quad \Phi(M) = \int_{p(M)}^{p(1)} \varphi(p)\,dp, \quad \varphi(p) = \frac{\operatorname{ctg}\alpha}{\rho V^2} \qquad (1.2)$$

where the dependence of φ on p and of p on M is derived from the conditions that the stream must be isoenergetic and isentropic, with allowance for the gas equations of state. For a perfect gas $\Phi(M)$ is defined in terms of elementary functions by the formula

$$\Phi(M) = \varepsilon^{-1/2} \operatorname{arctg}\sqrt{\varepsilon(M^2 - 1)} - \operatorname{arctg}\sqrt{M^2 - 1} = \pi(\varepsilon^{-1/2} - 1)/2 - 1/M(\varkappa - 1) + O(M^{-3}), \quad \varepsilon = (\varkappa - 1)/(\varkappa + 1) \qquad (1.3)$$

When $M_0 = 1$ the first term of the expansion of $\Phi(M)$ yields at the break point 2θ for the c^--characteristic which reaches the x-axis as $x \to \infty$. The curves calculated by (1.2) and (1.3) for various $\varkappa$ and $M_0 = 1.002$ are shown in Fig.2,a by solid lines (figures at these curves indicate the value of $\varkappa$). Similar curves obtained for the same M_0 but for r and ν not simultaneously zero, using numerical integration of (1.1) by the method of characteristics are shown in Fig.2,a by dash lines $(\nu = 0, r = 5)$ and in Fig.2,b $(\nu = 1$ by solid lines for $r = 0$ and for $r = 5$ by dash lines). The method of characteristics was, obviously, used for computations up to finite Mach numbers (from 30 to 50 depending on $\varkappa$). The respective curves for

$M \gg 1$ according to obtained results proved to be virtually linearly dependent on M^{-1} also when $\nu = r = 0$. This makes it possible to continue these up to $M^{-1} = 0$, i.e. to $M = \infty$ and thus determine $\theta(\infty, 1.002)$. Note that in all examples calculated for fixed $M \gg 1, \varkappa, r$ and $M_0 \simeq 1$ in the axisymmetric case θ is very close to half of the magnitude obtained for $\nu = 0$.

Below, we consider only isoenergetic and isentropic supersonic flows with $w \equiv 0$, for which all thermodynamic variables are functions of V only, and V and ϑ satisfy to quasilinear equations in partial derivatives. We pass, as in /1—6/, from V to the new variable η using the equality $V = (1 - \eta) V_m$, where V_m is the maximum flow velocity. The variable η defines the relative difference between V and its maximum value which decreases in Ω_i from $\eta = \Delta_i$ on Σ_i to zero as $M \to \infty$, with $\Delta_i > \Delta_{i+1}$, $i = 0, 1, \ldots$. For a perfect gas we have in the considered flows $M = (1 - \eta) \times [(\varkappa - 1)\eta(2 - \eta)/2]^{-1/2}$, hence for $M \gg (\varkappa - 1)^{-1/2}$

$$\eta = \frac{1}{(\varkappa - 1) M^2}\left[1 - \frac{9}{4(\varkappa - 1) M^2} + \ldots\right] \approx \frac{1}{(\varkappa - 1) M^2} \tag{1.4}$$

2. Owing to the parabolic degeneration occurring for $0 < \eta \leqslant \Delta_i \leqslant \Delta_0$, the flow in Ω_i is fully determined when a segment of the isobar Σ_i on which $\eta = \Delta_i \leqslant \Delta_0$ and $0 \leqslant \vartheta \leqslant \vartheta_{wi}$ where ϑ_{wi} is the value of ϑ at the upper point of Σ_i. From this and that for $\nu = 0$ the equations do not contain only x but, also, y, the expediency of changing the roles of the independent and dependent variables, i.e. passing to the hodograph plane $\eta\vartheta$, becomes evident. The problem then reduces (see, e.g., /7/) to the determination of the Legendre potential Φ in terms of its derivatives with respect to the hodograph variables expressed in terms of x and y. For a perfect gas the equation for $\Phi = \Phi(\eta, \vartheta)$ and formulas for x and y, with $\nu = 0$ and η and ϑ taken as the independent variables, assumes the form

$$\eta\chi(\eta, \varkappa)\Phi_{\eta\eta} - \Phi_{\vartheta\vartheta} + (1 - \eta)\Phi_\eta = 0 \tag{2.1}$$

$$x = -\Phi_\eta \cos\vartheta - (\Phi_\vartheta \sin\vartheta)/(1 - \eta), \quad y = -\Phi_\eta \sin\vartheta + (\Phi_\vartheta \cos\vartheta)/(1 - \eta)$$

$$\chi(\eta, \varkappa) = \frac{(\varkappa - 1)(2 - \eta)(1 - \eta)^2}{2(1 - \eta)^2 + (\varkappa - 1)(\eta - 2)\eta} = (1 - \varkappa)\left[1 + \left(\varkappa - \frac{3}{2}\right)\eta + O(\eta^2)\right]$$

where subscripts η and ϑ denote the respective partial derivatives of $\Phi(\eta, \vartheta)$. The first equation of system (2.1) and its simplified form in the case of small η, and, which, owing to the coefficient at $\Phi_{\eta\eta}$, approaching zero, confirms the indicated above parabolic degeneration, was apparently first used for explaining it in /1/. We shall show that (2.1) enables us to obtain more complete information about the solution in Ω_i.

Let us complement (2.1) by boundary conditions. These are based on that for $\eta = \Delta_i$ and $0 \leqslant \vartheta \leqslant \vartheta_{wi}$ we have $x(\eta, \vartheta) = X_i(\vartheta)$ and $y(\eta, \vartheta) = Y_i(\vartheta)$, and on the known functions $X_i(\vartheta)$ and $Y_i(\vartheta)$ obtained from the calculation of the near field, and, also, that $y(\eta, 0) = 0$ for $0 \leqslant \eta \leqslant \Delta_i$. We omit in what follows, as a rule, the subscript «i» and, taking into account (2.1) rewrite these conditions in the form

$$\Phi_\eta(\Delta, \vartheta) = f_1(\vartheta), \quad \Phi_\vartheta(\Delta, \vartheta) = f_2(\vartheta) \quad \text{for } 0 \leqslant \vartheta \leqslant \vartheta_w \tag{2.2}$$

$$\Phi_\vartheta(\eta, 0) = 0 \quad \text{for} \quad 0 \leqslant \eta \leqslant \Delta$$

where $f_j(\vartheta)$ are expressed in terms of $X(\vartheta)$ and $Y(\vartheta)$ in conformity with the second and third of equalities (2.1), with $f_2(0) = 0$.

Solution of the first of Eqs. (2.1) with boundary conditions (2.2) may be sought by the method of separation of variables in the form

$$\Phi(\eta, \vartheta) = \sum_{m=0}^{\infty} \Phi_{1m}(\eta)\Phi_{2m}(\vartheta) \tag{2.3}$$

The absence in this problem of any conditions at the upper boundary of Ω makes it possible to choose for $\Phi_{2m}(\vartheta)$ any complete system of functions each of which satisfies an ordinary second order differential equation obtained by the substitution of (2.3) into (2.1), and the third of conditions (2.2). One of such systems is the sequence $1, \cos\vartheta, \cos 2\vartheta, \ldots$. Second order ordinary differential equations are also obtained for function $\Phi_{1m}(\eta)$ whose coefficients are variable, unlike the constant coefficients of $\Phi_{2m}(\vartheta)$. The point $\eta = 0$ is for these functions a regular singular point in whose neighborhood the solution can be generally represented as the sum of two generalized power series /8/. Exponents $\gamma_{1,2}$ of these series are, apart the dependence on m, $\gamma_1 = 0$ and $\gamma_2 = \gamma = (\varkappa - 2)/(\varkappa - 1) = 1 - \lambda$, where γ is expressed in terms of $\lambda = (\varkappa - 1)^{-1}$ for simplifying subsequent formulas. The above reasoning implies that for $\gamma \neq 0$, i.e. for $\varkappa \neq 2$, as well as for γ not any integer (for $\varkappa > 1$, $\varkappa = 1 + n^{-1}$, with $n = 2, 3, \ldots$ correspond to these), the solution is of the form

$$\Phi(\eta, \vartheta) = \chi_1(\eta, \vartheta) + \eta^\gamma \chi_2(\eta, \vartheta) \tag{2.4}$$

Here and subsequently, unless otherwise stated, $\chi_j(\eta, \vartheta)$ are assumed to be analytic functions of η. Below, these functions will be usually written without the arguments η and ϑ. Functions χ_1 and χ_2 are analytic also with respect to ϑ, which is actually immaterial in this case, since for their determination using conditions (2.2) it is not necessary to expand χ_j in ϑ. On the strength of the last of conditions (2.2) we have $\chi_{1\vartheta}(\eta, 0) = \chi_{2\vartheta}(\eta, 0) = 0$. The case of $\varkappa = 1 + n^{-1}$ with $n = 1, 2, \ldots$ will be briefly considered later.

In conformity with (2.1) and (2.4) we have

$$\eta^\lambda x = -\gamma\chi_2 \cos\vartheta + \eta\chi_3 + \eta^\lambda\chi_4, \quad \eta^\lambda y = -\gamma\chi_2 \sin\vartheta + \eta\chi_5 + \eta^\lambda\chi_6 \tag{2.5}$$

where $\chi_3 \div \chi_6$ is expressed in terms of derivatives of χ_1 and χ_2 with respect to η and ϑ, and $\chi_5(\eta, 0) = \chi_6(\eta, 0) = 0$. We eliminate from these equations $\gamma\chi_2$ and obtain the equality

$$y = x \operatorname{tg}\vartheta + \eta^{-\gamma}\chi_7 + \chi_8 \tag{2.6}$$

in which χ_7 and χ_8, as well as χ_5 and χ_6 vanish when $\vartheta = 0$. The last formula enables us to determine $\partial\vartheta / \partial\eta$ along a streamline and, to estimate the variation of ϑ along it from Σ to $x \to \infty$. For this we determine the total derivatives of both parts of (2.6) with respect to η, taking into account that along the streamline $dy / \partial x = \operatorname{tg}\vartheta$, eliminate x from the obtained equality using (2.5), we define the sought derivative by the resulting equation, and obtain

$$d\vartheta / \partial\eta = (\chi_9 + \eta^\lambda\chi_{10}) / (\chi_{11} + \eta^\lambda\chi_{12}) \tag{2.7}$$

Hence $\vartheta = \vartheta_\infty + \alpha\eta + o(\eta)$, where ϑ_∞ is the value approached by ϑ along the streamline as $x \to \infty$, α is a constant, and the total variation of ϑ is of order $O(\Delta)$, which agrees with the similar conclusions in /1—6/, but by no means implies the rectilinearity of streamlines asserted there. This follows from that in the integration, on fairly large interval x, of the streamlines equations $dy/dx = \operatorname{tg}\vartheta = \operatorname{tg}\vartheta_\infty + \beta\eta + o(\eta)$ in which $\beta = \alpha/\cos^2\vartheta_\infty$ is a small component, the contribution of the term $\beta\eta$ can be of any magnitude. That this is so for $\gamma < 0$ or $\varkappa < 2$ follows from (2.6), since in conformity with (2.6) and (2.5)

$$y = x \operatorname{tg}\vartheta_\infty + (x + \chi_{13})^{2-\varkappa}\chi_{14} + \chi_{15} \tag{2.8}$$

and $\chi_{14}(\eta, 0) = \chi_{15}(\eta, 0) = 0$. Thus, when $\varkappa < 2$, the streamlines for $x \to \infty$ noncoincident with the x-axis diverge to any extent from straight lines $y = x \operatorname{tg}\vartheta_\infty + \text{const}$. The assertion of streamline rectilinearity is basic to the solution derived in /1—6/. The disregard of this property is the reason of the difference between that solution and the one obtained here. We shall prove this for $M \gg (\varkappa - 1)^{-1/2}$, when in conformity with (1.4) $\eta \sim M^{-2}$ and, moreover $\eta^\lambda = \eta^{1/(\varkappa-1)} \sim \rho V$. Taking these factors into consideration, we retain in $\chi_2 \div \chi_6$ from (2.5) only the zero terms of expansion in η. After some obvious transformations, equalities (2.5) assume the form

$$\begin{aligned}\rho V x &= A(\vartheta)\cos\vartheta + \eta^{1/(\varkappa-1)} C(\vartheta) + \eta B(\vartheta) + o(\eta, \eta^{1/(\varkappa-1)}) = \\ &A(\vartheta)\cos\vartheta + \rho V X(\vartheta) + (\rho V)^{\varkappa-1} B^\circ(\vartheta) + o(M^{-2}, M^{-2/(\varkappa-1)}) \\ \rho V y &= A(\vartheta)\sin\vartheta + \eta^{1/(\varkappa-1)} H(\vartheta) + \eta D(\vartheta) + o(\eta, \eta^{1/(\varkappa-1)}) = \\ &A(\vartheta)\sin\vartheta + \rho V Y(\vartheta) + (\rho V)^{\varkappa-1} D^\circ(\vartheta) + o(M^{-2}, M^{-2/(\varkappa-1)})\end{aligned} \tag{2.9}$$

where, owing to symmetry with respect to ϑ functions $A(\vartheta)$, $B(\vartheta)$, $C(\vartheta)$, $X(\vartheta)$, and $B^\circ(\vartheta)$ are even, and the remaining odd.

The equations of streamlines are obtained from this with the same accuracy when ϑ is replaced by ϑ_∞ or ϑ_i which is the value of ϑ at the point of issue of a streamline defined by curve Σ_i. The first pairs of terms at the right-hand sides of (2.9) yield the solution obtained in /1—6/. But the second terms there dominate the next following ones only for $\varkappa > 2$, while for $\varkappa < 2$ the third terms which are absent in /1—6/ predominate (the case of $\varkappa > 5/3$ is of interest in some not purely gasdynamic applications, e.g., in the theory of shallow waters that correspond to $\varkappa = 2$). We stress that according to the analysis presented below, the neglect of the second and third terms in (2.9) is usually admissible (but not in equations of streamlines) only at Mach numbers exceeding several tens. Finally, we would point out that according to a reasoning similar to that used for ordinary differential equations (see /8/), the formulas for $\varkappa = 2$ differ from the obtained above by the substitution in the left-hand sides of (2.5) of η and in the right-hand sides of (2.5) and (2.7) of $\eta \ln \eta$, respectively, for η^λ; the substitution of $\ln x$ for the power of complex $2 - \varkappa$ in (2.8); also the substitution of $\ln \eta$ in (2.6) for $\eta^{-\gamma}$, and, finally, the substitution of $\eta \ln \eta$, $M^{-2} \ln M$ and $\rho V \ln (\rho V)$ in (2,9), respectively for $\eta^{1/(\varkappa-1)}$, $M^{-2/(\varkappa-1)}$ and $(\rho V)^{\varkappa-1}$. Moreover, when $\varkappa = 2$, χ_{14} and χ_{15} are finite but not analytic functions of η in the vicinity of point $\eta = 0$. For $\varkappa = 1 + n^{-1}$ with $n \geqslant 2$ similar substitutions are effected in higher and higher terms, and this makes these formulas valid as regards their principal terms.

The accuracy of the distant field determination by the above formulas can be achieved, as already indicated, in two ways. The first of these consists of extending the near field with the expansion of (2.5) in η limited to a fixed number k of terms, for instance, two or three (any numerically calculated part of the flow is called here the near field). Although this way seems to be the most natural, formulas (2.5) make it possible to attain the required accuracy also in the case of fixed dimensions of the near field by increasing k. In both cases the $2k$ coefficients dependent on ϑ can be obtained by isolating in the near field k isolines $\eta = \Delta_i \leqslant \Delta_0$, i.e. by constructing $2k$ functions $X_i(\vartheta)$ and $Y_i(\vartheta)$, $i = 1, \ldots, k$. By defining along these lines the respective intervals of expansions (2.5) we obtain $2k$ linear algebraic equations for the determination of the equal number of unknown coefficients dependent on ϑ.

Actually, when carrying out the above procedure, it is by no means necessary to use isobars that have open regions of determination, i.e. those to the right of Σ_0. The isobars to the left of Σ_0 may prove suitable for this purpose, provided they belong to the same analytically homogeneous flow region as those on the right of it. In many instances the required smoothness is retained, if not up to subsonic, then to fairly moderate supersonic velocities (e.g., for $r \equiv \text{const}$ and plane transition surface to $M = 1$). The above makes at least not only possible to try to use moderately supersonic isobars but, also, to count on the validity of obtained formulas to the left of Σ_0, although possibly not down to $M = 1$.

3. The Legendre transform naturally yields the previous formulas for x and y from (2.1) also in the axisymmetric case. However the equation for $\Phi(\eta, \vartheta)$ is in this case not so simple. This is due to that now the continuity equation

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \nu \frac{\rho v}{y} = 0$$

has a free term containing y which inhibits linearization in variables of the hodograph. This difficulty can, however, be avoided, at least in the proximity of the axis of symmetry where $\rho v / y \simeq \partial(\rho v) / \partial y$. Note that the region in which such substitution is valid, although remaining narrow, as also Ω_0, in the sence of the ratio of dimensions relative to y and x can widen arbitrarily as $x \to \infty$. For $\nu = 1$ the first of Eqs. (2.1) is replaced in that region by

$$2\eta\chi(\eta, \varkappa)\Phi_{\eta\eta} - \Phi_{\vartheta\vartheta} + (1 - \eta)\Phi_\eta - \chi^\circ(\eta, \varkappa)\sin 2\vartheta\,[(1 - \eta)\Phi_{\eta\vartheta} + \Phi_\vartheta] = 0 \qquad (3.1)$$

$$\chi^\circ(\eta, \varkappa) = [1 + \varkappa\eta(\eta - 2)] / [2(1 - \eta)^2 + (\varkappa - 1)\eta(\eta - 2)]$$

where function $\chi(\eta, \vartheta)$ is the same as in (2.1). We seek a solution of (3.1), as in the case of (2.1) in the form of the sum of two solutions of the form $\eta^{\gamma_i}\chi_i(\eta, \vartheta)$ with beforehand unknown exponents γ_i and functions of χ_i analytic in the neighborhood of point $\eta = \vartheta = 0$. Since the expansion in $\sin\vartheta$ begins with the first power of ϑ the equation for γ_i which is obtained by equating to zero in each solution the coefficients at the lowest power of η and ϑ, i.e. at $\eta^{\gamma_i - 1}\vartheta^0 \equiv \eta^{\gamma_i - 1}$ yields $\gamma_1 = 0$ and $\gamma_2 = \gamma = 1 - \lambda$ with $\lambda = 1 / [2(\varkappa - 1)]$. Thus for $\lambda \neq 1, -1, -2, \ldots,$ which corresponds to $\varkappa \neq 1 + 1/(2n)$ the solution of (3.1) is of the form (2.4). When $\varkappa = 3/2$ the multiplier η^γ in the second term in (2.4) must be replaced by $\eta \ln \eta$. All of the above remains valid when any function of η and ϑ analytic in the neighborhood of point $\eta = \vartheta = 0$ is substituted in (3.1) for $\chi^\circ \sin 2\vartheta$. The latter can be readily proved by taking into account that by virtue of the third of Eqs. (2.1) $\chi_{i\eta\vartheta} = \chi_{i\vartheta} = 0$ when $\vartheta = 0$. This remark opens way for proving the validity of the reasoning in Sect.3 everyhwhere in Ω_i and not only near the x-axis. Hence by rewriting the above formulas in the form $\rho V R^2 = A_0^2(1 + \delta\vartheta^2)$, where A_0 and δ are constants, and R is the distance from the coordinate origin, valid in the vicinity of $\eta = 0$, we obtain with an accuracy to ϑ^2 for $\rho v / y$

$$\frac{\rho v}{y} = \frac{1}{1 - (3 - \delta)\sin^2\vartheta}\frac{\partial(\rho v)}{\partial y}$$

The alterations that must be introduced in this case in (3.1) do not affect γ_1 and γ_2, and their effect on functions $\chi_i(\eta, \vartheta)$ is immaterial for the subsequent analysis. The validity of solution (2.4) with the obtained above values of γ and functions χ_1 and χ_2 analytic in the neighborhood of point $\eta = \vartheta = 0$ not only in the small vicinity of the x-axis is supported by the results of calculations presented in Sect.4 and, also, by that the indicated solution formally satisfies the complete nonlinear equation for $\Phi(\eta, \vartheta)$.

Omitting the detailed analysis, which is analogous to that carried out for $\nu = 0$, we present the final relations. Formulas (2.5)—(2.7) apply to the axisymmetric case without any change, except that now λ is equal $1 / [2(\varkappa - 1)]$ and not its double value, as in the case of $\nu = 0$. Taking into account the distinction in λ and $\gamma = 1 - \lambda$, the analog of Eq. (2.8) assumes the form

$$y = x \operatorname{tg} \vartheta_\infty + (x + \chi_{13})^{3 - 2\varkappa}\chi_{14} + \chi_{15}$$

and formulas (2.9) becomes

$$\begin{aligned}\sqrt{\rho V}x &= A(\vartheta)\cos\vartheta + \eta^{1/[2(\varkappa-1)]}C(\vartheta) + \eta B(\vartheta) + o(\eta, \eta^{1/[2(\varkappa-1)]}) = \\ &A(\vartheta)\cos\vartheta + \sqrt{\rho V}X(\vartheta) + (\rho V)^{\varkappa-1}B^\circ(\vartheta) + o(M^{-2}, M^{-1/[(\varkappa-1)]}) \\ \sqrt{\rho V}y &= A(\vartheta)\sin\vartheta + \eta^{1/[2(\varkappa-1)]}H(\vartheta) + \eta D(\vartheta) + \\ &o(\eta, \eta^{1/[2(\varkappa-1)]}) = A(\vartheta)\sin\vartheta + \sqrt{\rho V}Y(\vartheta) + (\rho V)^{\varkappa-1}D^\circ(\vartheta) + o(M^{-2}, M^{-1/[(\varkappa-1)]})\end{aligned} \qquad (3.2)$$

For $\varkappa = {}^3/_2$ the power terms in respective formulas are replaced by $\eta \ln \eta, \ln \eta, \ln x, M^{-2} \ln M$, and $\rho V \ln(\rho V)$ in conformity with the rules analogous to those used in Sect.2.

According to the derived relationships, streamlines may be arbitrarily close to straight lines, and formulas (3.2) for each of them, i.e. for fixed ϑ, in principal orders reduce to a flow from a spherical source of intensity $A(\vartheta)$ with its pole at point $x = X(\vartheta), y = Y(\vartheta)$ only when $\varkappa > {}^3/_2$. Otherwise contrary to /1—6/ streamlines arbitrarily strongly deviate from the straight lines as $y = x \operatorname{tg} \vartheta_\infty + \text{const}$ when $x \to \infty$, and the terms proportional to $B(\vartheta)$ and $D(\vartheta)$ or to $B^\circ(\vartheta)$ and $D^\circ(\vartheta)$ become the principal terms in (3.2) in comparison with those that define the pole transfer. It ought to be stressed, however, that for the majority of practically interesting $\varkappa$ the difference between $\varkappa - 1$ and 1/2 is not as great as from unity. Hence for $\nu = 1$ the relative dominant part of the second or third terms (respectively for $\varkappa > {}^3/_2$ and $\varkappa \leqslant {}^3/_2$) is not so pronounced as in the plane case. In such situations it is expedient to retain in formulas (3.2) all three first terms.

4. To ascertain the accuracy of the established here laws of free expansion flows, extensive calculations of plane and axisymmetric streams were carried out. Some of these were already used for the construction of Fig.2. A special program based on the so-called "direct" scheme of the method of characteristics and which is a modification of the program composed earlier /9/ was used for computations. The above modification ensured a very rapid and exact computation of super- and hypersonic streams. Thus the determination of the acceleration section of a plane stream from $M_0 = 1.002$ with $\varkappa = 1.25$ to $M = 30$ or to $x = 2.7 \cdot 10^7$ with over-all errors of the flow rate and momentum not exceeding 1% required 18 min. of computer time.

Since further calculations relate to the x-axis, we present the formulas which are obtained for $\vartheta = 0$ after some transformations and rejection of higher order terms in the right-hand sides of the first equalities in (2.9) and (3.2). For $\nu = 0$ according to (2.9) we have on the x-axis

$$\rho V x = A + CM^{-2/(\varkappa-1)} + \begin{cases} BM^{-2} & \text{for } \varkappa \neq 2 \\ BM^{-2} \ln M & \text{for } \varkappa = 2 \end{cases} \tag{4.1}$$

Here and subsequently A, B and C are functions of $\nu, \varkappa, M_0$, and r. In the case of axisymmetric flow the respective relations are of the form

$$\sqrt{\rho V} x = A + CM^{-1/(\varkappa-1)} + \begin{cases} BM^{-2} & \text{for } x \neq {}^3/_2 \\ BM^{-2} \ln M & \text{for } \varkappa = {}^3/_2 \end{cases} \tag{4.2}$$

The results of numerical computations for plane streams are shown in Fig.3 by solid lines, with the numerals 1—5 denoting curves related to $M_0 = 1.002$, $r = 0$ and correspondingly to $\varkappa = 1.25, {}^4/_3, {}^7/_5, {}^5/_3$, and 2. Curves 1°–5° relate to $r = 5$ and the same M_0 and $\varkappa$. Besides the solid lines two approximating curves are plotted for each case, viz. dash and dash-dot lines which correspond, respectively to computations using two or all three terms in the right-hand side of (4.1). When $\varkappa = 1.25$ the dash-dot and the solid lines merge in Fig.3. The coefficients A, B and C were determined using the condition of congruence with dash lines at points $M = M_m$ and 10, and $M = M_m$, 10 and 6 with the dash and dash-dot lines, respectively (M_m is the maximum value of M calculated for a particular case). The curves denoted by numerals 6 and 7 relate to discharge with $\varkappa = 1.4$ and $r = 0$ of plane streams at $M_0 = 3$ and 4. Here, x was measured from the outlet cross section but was normalized, as everywhere in this paper, with respect to the half-height of the nozzle minimum cross section (the stream at that cross section was assumed uniform and $M \equiv 1$). The results of similar computations and their comparison with (4.2) in the case of axisymmetric flows are plotted in Fig.4 in basically the same layout as in Fig.3. The difference between these two figures is primarily in that curves 6 and 6° in Fig.4 relate to $M_0 = 1.002$ and $\varkappa = 3$. Moreover, Fig.4 does not contain all dash-dot lines but only those most distinct from the solid lines. The validity of laws established in Sects. 2 and 3 is demonstrated in Figs.3 and 4, which also show then, as a rule, extremely slow transition to asymptotic formulas as $M \to \infty$, which are obtained when only the first terms are retained in the right-hand sides of respective formulas.

We note in conclusion that strictly speaking the analysis in Sect.3 is valid only in narrow open zones which appear at very high M, and in which the simplification of the continuity equation used in Sect.3 is justified. In spite of this, even with allowance for the remark at the end of Sect.3, the efficiency of formulas (4.2) demonstrated in Fig.4 makes it possible to expect that the laws established in Sect.3 are valid not only in such zones, but also in a considerable part of the free expansion region.

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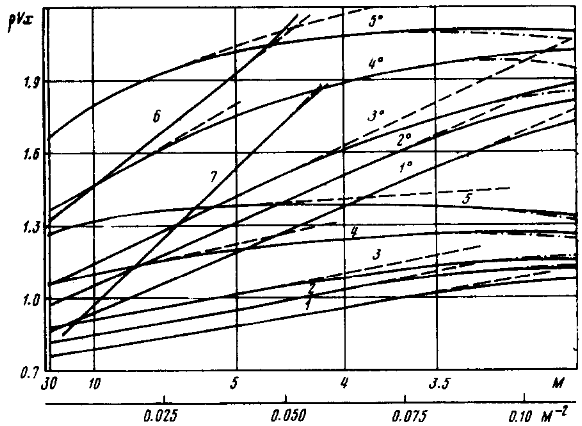

Fig.3

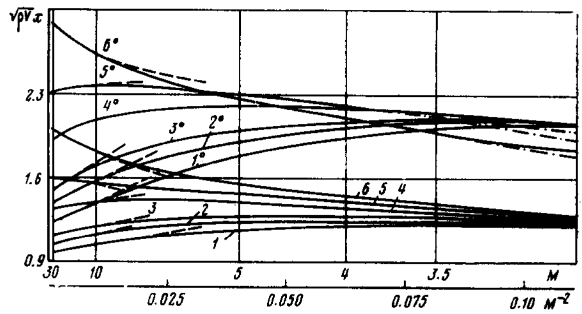

Fig.4

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